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Testing $f(R)$-theories using the first time derivative of the orbital period of the binary pulsars

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ABSTRACT

In this paper we use one of the Post-Keplerian parameters to obtain constraints on $f(R)$-theories of gravity. Using Minkowskian limit, we compute the prediction of $f(R)$-theories on the first time derivative of the orbital period of a sample of binary stars, and we use our theoretical results to perform a comparison with the observed one. Selecting a sample of relativistic binary systems we estimate the parameters of an analytic $f(R)$-gravity. We find that the theory is not ruled out if we consider only the double neutron star systems, and in this case we can cover the existing gap between the General Relativity prediction and the observed data.

Key words: gravitation – binary pulsar systems.

1 INTRODUCTION

The gravitational waves (GWs) are one of the most promising tools to study astrophysical systems like Neutron Stars (NS), coalescing binary systems, Black Holes (BHs), and White Dwarfs (WDs).

The observational indirect evidences of gravitational radiation were measured on the system B1913+16, known as the Hulse-Taylor binary pulsar, and confirmed in others relativistic binary systems. The prediction of General Relativity (GR) on the first time derivative of the orbital period in binary pulsar systems was studied by Hulse and Taylor (1975) and Weisberg et al. (2010), for which the discrepancy on observed data with respect to the prediction is $\sim 1\%$. However, the observational results should be explained using a different formulation of gravity (Freire et al. 2012). As shown in De Laurentis and Capozziello (2011), these systems could represent a good test for Extended Theories of Gravity (ETG). Considering a class of analytic $f(R)$-theories, it is possible evaluate the gravitational radiated power in weak field limit. In this approximation we find that the energy radiated depends on the third derivative of the quadrupole, as predicted by GR, and the fourth derivative representing the corrective contribution to the theory. This result can be used to set constraints on the theory, comparing the prediction on the first time derivatives of the orbital period with respect to the observed one. The outline of the paper is the following: in sec. 2 we briefly introduce the weak field limit approximation of $f(R)$-theories of gravity. In sec. 3 we apply the theoretical results previously obtained to binary systems computing the energy lost through GWs emission. In sec. 4 we compute the first time derivative of the orbital period in $f(R)$-theories of gravity, and we compare the theoretical prediction with the observed data. Finally, in sec. 5 we give our conclusions and remarks.

2 F(R)-GRAVITY BACKGROUND

The $f(R)$-theories are based on corrections and enlargements of the GR theory adding higher-order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of quantum gravity (Capozziello and De Laurentis 2011).

Starting from the following field equations in $f(R)$-gravity (looking for major details at Capozziello and De Laurentis (2011), Nojiri and Odintsov (2011), Nojiri and Odintsov (2007), Capozziello and Francaviglia (2008), Capozziello et al.

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(2009))[1]:

$$f'(R)R_{\mu\nu}-\frac{f(R)}{2}g_{\mu\nu}-f'(R)_{;\mu\nu}+g_{\mu\nu}\Box_g f'(R) = \frac{\mathcal{X}}{2}T_{\mu\nu}\,, \quad (1)$$

$$3\Box f'(R)+f'(R)R-2f(R) = \frac{\mathcal{X}}{2}T\,, \quad (2)$$

the Minkowskian limit can be calculate for a class of analytic $f(R)$-Lagrangian (*i.e.* Taylor expandable in term of the Ricci scalar[2])

$$f(R)=\sum_n\frac{f^n(R_0)}{n!}(R-R_0)^n\simeq f_0+f'_0R+\frac{f''_0}{2}R^2+\dots\,. \quad (3)$$

At the first order, in term of the perturbations, the field equations become

$$f'_0\left[R^{(1)}_{\mu\nu}-\frac{R^{(1)}}{2}\eta_{\mu\nu}\right]-f''_0\left[R^{(1)}_{,\mu\nu}-\eta_{\mu\nu}\Box R^{(1)}\right]=\frac{\mathcal{X}}{2}T^{(0)}_{\mu\nu}\,, \quad (4)$$

where $f'_0=\frac{df}{dR}\Big|_{R=0}$, $f''_0=\frac{d^2f}{dR^2}\Big|_{R=0}$ and $\Box={}_{,\sigma}{}^{,\sigma}$ that is d'Alembert operator. Here, the Ricci tensor and scalar read

$$\begin{cases} R^{(1)}_{\mu\nu}=h^{\sigma}_{(\mu,\nu)\sigma}-\frac{1}{2}\Box h_{\mu\nu}-\frac{1}{2}h_{,\mu\nu} \\ \\ R^{(1)}=h_{\sigma\tau}{}^{,\sigma\tau}-\Box h \end{cases} \quad (5)$$

Now, assuming that the source is localized in a finite region, as a consequence outside this region $T_{\mu\nu}=0$ and

$$R^{(1)}_{\mu\nu}=\Box h_{\mu\nu}=0\,. \quad (6)$$

From here it is possible calculate the energy momentum tensor of gravitational field in $f(R)$-gravity, adopting the definition given in Landau and Lifshitz (1962) and De Laurentis and Capozziello (2011), so that it satisfies a conservation law as required by the Bianchi identities:

$$\begin{aligned} t^\lambda_\alpha &= \frac{1}{\sqrt{-g}}\left[\left(\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda}}-\partial_\xi\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}\right)g_{\rho\sigma,\alpha}+\right.\\ &\left.+\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}-\delta^\lambda_\alpha\mathcal{L}\right]\,. \end{aligned} \quad (7)$$

Starting from above equation, De Laurentis and Capozziello (2011) have shown that the energy momentum tensor consists of a sum of a GR contribution plus a term coming from $f(R)$-gravity

$$t^\lambda_\alpha=f'_0t^\lambda_{\alpha|\mathrm{GR}}+f''_0t^\lambda_{\alpha|_{f(R)}}\,, \quad (8)$$

that in term of the perturbation h is

$$\begin{aligned} t^\lambda_\alpha \sim\ & f'_0t^\lambda_{\alpha|\mathrm{GR}}+f''_0\{(h^{\rho\sigma}{}_{,\rho\sigma}-\Box h)\left[h^{\lambda\xi}{}_{,\xi\alpha}-h^{,\lambda}{}_{\alpha}-\right.\\ &\left.+\frac{1}{2}\delta^\lambda_\alpha(h^{\rho\sigma}{}_{,\rho\sigma}-\Box h)\right]-h^{\rho\sigma}{}_{,\rho\sigma\xi}h^{\lambda\xi}{}_{,\alpha}+\\ &+h^{\rho\sigma}{}_{,\rho\sigma}{}^{\lambda}h_{,\alpha}+h^{\lambda\xi}{}_{,\alpha}\Box h_{,\xi}-\Box h^{,\lambda}h_{,\alpha}\} \end{aligned} \quad (9)$$

To simplify the above equation the weak field limit approximation is taken into account, *i.e.* the source $h_{\mu\nu}$ will be written as function of a single scalar variable $t'=t-r$, and as a consequence, it will be almost plane (Maggiore 2007; De Laurentis and Capozziello 2011).

Finally, the energy momentum tensor assumes the following form

$$t^\lambda_\alpha=\underbrace{f'_0k^\lambda k_\alpha\left(\dot{h}^{\rho\sigma}\dot{h}_{\rho\sigma}\right)}_{GR}-\underbrace{\frac{1}{2}f''_0\delta^\lambda_\alpha\left(k_\rho k_\sigma\ddot{h}^{\rho\sigma}\right)^2}_{f(R)}\,. \quad (10)$$

To be more precise, the first term, depending on the choice of the constant f'_0, is the standard GR term, the second is the $f(R)$ contribution. It is worth noticing that the order of derivative is increased of two degrees consistently to the fact that $f(R)$-gravity is of fourth-order in the metric approach (De Laurentis and Capozziello 2011).

3 RADIATED ENERGY

In order to calculate the radiated energy of a gravitational waves source, De Laurentis and Capozziello (2011) suppose that $h_{\mu\nu}$ can be represented by a discrete spectral representation. The periodicity T will be proportional to the inverse of the difference of the pair of frequency components in the wave, and then, the average of $\frac{dE}{dt}$ must be evaluated over an interval equal to or greater than T (Landau and Lifshitz 1962; Maggiore 2007). The instantaneous flux of energy through a surface of area $r^2d\Omega$ in the direction $\hat{x}$ is given by

$$\frac{dE}{dt}=r^2d\Omega\hat{x}^it^{0i}\,, \quad (11)$$

and the average flux of energy can be written as

$$\left\langle\frac{dE}{dt}\right\rangle=r^2d\Omega\hat{x}^i\langle t^{0i}\rangle\,. \quad (12)$$

Defining the following moments of the mass-energy distribution:

$$M(t)\simeq\int d^3\vec{x}\,T^{00}(\vec{x},t)\,, \quad (13)$$

$$D^k(t)\simeq\int d^3\vec{x}\,x^kT^{00}(\vec{x},t)\,, \quad (14)$$

$$Q^{ij}(t)\simeq\int d^3\vec{x}\,x^ix^jT^{00}(\vec{x},t)\,, \quad (15)$$

and analyzing the radiation in terms of multipoles, De Laurentis and Capozziello (2011) found

[1] $T_{\mu\nu}=\frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the energy momentum tensor of matter (T is the trace), $\mathcal{X}=\frac{16\pi G}{c^4}$ is the coupling, $f'(R)=\frac{df(R)}{dR}$, $\Box_g={}_{;\sigma}{}^{;\sigma}$, and $\Box={}_{,\sigma}{}^{,\sigma}$. We adopt a $(+,-,-,-)$ signature, and indicate with "," partial derivative and with " ;" covariant derivative with regard to $g_{\mu\nu}$; all Greek indices run from $0,...,3$ and Latin indices run from $1,...,3$; g is the determinant.
[2] For convenience we will use f instead of $f(R)$. All considerations are developed here in metric formalism. From now on we assume physical units $G=c=1$.

$$\left\langle t_\alpha^\lambda\right\rangle = \left\langle f_0' k^\lambda k_\alpha \frac{4}{r^2}\left[\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right)^2 - 2\left(\hat{x}_k\dddot{Q}^{ik}\right)\left(\hat{x}_j\dddot{Q}^{ij}\right) + \right.\right.$$
$$\left.+\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)\right] - f_0''\delta_\alpha^\lambda\left(k_\rho k_\sigma\right)^2\frac{2}{r^2}\left[\left(\hat{x}_i\hat{x}_j\ddddot{Q}^{ij}\right)^2 + \right.$$
$$\left.\left. -2\left(\hat{x}_k\ddddot{Q}^{ik}\right)\left(\hat{x}_j\ddddot{Q}^{ij}\right) + \left(\ddddot{Q}^{ij}\ddddot{Q}_{ij}\right)\right]\right\rangle . \tag{16}$$

Using the result in eq. (12) and integrating over all directions they computed the total average flux of energy due to the tensor wave,

$$\underbrace{\left\langle\frac{dE}{dt}\right\rangle}_{(total)} = \frac{G}{60}\left\langle\underbrace{f_0'\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)}_{GR} - \underbrace{f_0''\left(\ddddot{Q}^{ij}\ddddot{Q}_{ij}\right)}_{f(R)}\right\rangle . \tag{17}$$

Precisely, for $f_0'' \to 0$ and $f_0' \to \frac{4}{3}$, eq. (17) becomes

$$\underbrace{\left\langle\frac{dE}{dt}\right\rangle}_{(GR)} = \frac{G}{45}\left\langle\dddot{Q}^{ij}\dddot{Q}_{ij}\right\rangle , \tag{18}$$

which is the well-known result of GR (Landau and Lifshitz 1962; Weinberg 1972). An important remark is related to the absence of monopole and dipole terms in our considerations. In our case, all the calculations are performed in the Jordan frame so$f(R)$-gravity results as a mere extension of GR being $f(R) = R$, so any dipole terms is null (as shown in Will (1993) in table 10.2). In order to put in evidence such contributions, we have to pass in the Einstein frame where the additional degrees of freedom of gravitational field can be recast in term of scalar fields. In this case, monopole and dipole terms explicitly come out (Will 1993; Naef and Jetzer 2011; Damour and Esposito-Farese 1996). The two approaches are conformally equivalent but in the Einstein frame monopole and dipole terms can come out (see, e.g. Capozziello and De Laurentis (2011)).

4 APPLICATION TO PULSAR BINARY SYSTEMS

Now, our goal is to use a sample of binary pulsar systems to fix bounds on $f(R)$ parameters. To do this, we assume that the motion is Keplerian and the orbit is in the (x, y)-plane. We define m_p as the pulsar mass, m_c as the companion mass, and $\mu = \dfrac{m_c m_p}{m_c + m_p}$ as the reduced mass. In (x, y)-plane the quadrupole matrix is

$$Q_{ij} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ij} , \tag{19}$$

where i, j are the indexes in the orbital plane, r is the equation of the elliptic Keplerian orbit and ψ is eccentric anomaly, and both of them are time dependent.

To compute the radiated power we need the third and fourth derivatives of quadrupole, so we must compute the time derivatives using the following relation given in Maggiore (2007)

$$\dot{\psi} = \left(\frac{Gm_c}{a^3}\right)^{\frac{1}{2}}\left(1-\epsilon^2\right)^{-\frac{3}{2}}\left(1+\epsilon\cos\psi\right)^2 , \tag{20}$$

where a is the semi-major-axis, and ϵ is the orbital eccentricity. We obtain the following relations for the time derivatives of the quadrupole

$$\dddot{Q}_{11} = \mathcal{H}_1 \sin 2\psi(\epsilon\cos\psi + 1)^2(3\epsilon\cos\psi + 4), \tag{21}$$

$$\dddot{Q}_{22} = -\mathcal{H}_1(8\cos\psi + \epsilon(3\cos 2\psi + 5)) \times \times \sin\psi(\epsilon\cos\psi + 1)^2, \tag{22}$$

$$\dddot{Q}_{12} = -\mathcal{H}_1(\epsilon\cos\psi + 1)^2 \times \times(5\epsilon\cos\psi + 3\epsilon\cos 3\psi + 8\cos 2\psi), \tag{23}$$

$$\ddddot{Q}_{11} = \mathcal{H}_2\left[15\epsilon^2\cos 4\psi + 50\epsilon\cos 3\psi + \right. \left. + \left(12\epsilon^2 + 32\right)\cos 2\psi + 6\epsilon\cos\psi - 3\epsilon^2\right] , \tag{24}$$

$$\ddddot{Q}_{22} = -\mathcal{H}_2\left[15\epsilon^2\cos 4\psi + 50\epsilon\cos 3\psi + \right. \left. + \left(24\epsilon^2 + 32\right)\cos 2\psi + 14\epsilon\cos\psi - 7\epsilon^2\right] , \tag{25}$$

$$\ddddot{Q}_{12} = 2\mathcal{H}_2\sin\psi\left[15\epsilon^2\cos 3\psi + 50\epsilon\cos 2\psi + \right. \left. + \left(33\epsilon^2 + 32\right)\cos\psi + 30\epsilon\right] , \tag{26}$$

where

$$\mathcal{H}_1 = \frac{(2\pi)^{5/3}\,G^{2/3}\,m_c\,m_p}{T^{5/3}\left(1-\epsilon^2\right)^{5/2}\,\sqrt[3]{m_c+m_p}} ,$$

$$\mathcal{H}_2 = \frac{2^{2/3}\pi^{8/3}G^{2/3}m_c\,m_p\left(\epsilon\cos\psi + 1\right)^3}{T^{8/3}\left(\epsilon^2-1\right)^4\,\sqrt[3]{m_c+m_p}} .$$

Now, from eq. (17), we can perform the time average of the radiated power writing

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{1}{T}\int_0^T dt\frac{dE(\psi)}{dt} = \frac{1}{T}\int_0^{2\pi}\frac{d\psi}{\dot{\psi}}\frac{dE(\psi)}{dt} , \tag{27}$$

and finally we get the first time derivative of the orbital period

$$\dot{T}_b = -\frac{3}{20}\left(\frac{T}{2\pi}\right)^{-\frac{5}{3}}\frac{\mu G^{\frac{5}{3}}(m_c+m_p)^{\frac{2}{3}}}{c^5(1-\epsilon^2)^{\frac{7}{2}}} \times \times\left[f_0'\left(37\epsilon^4 + 292\epsilon^2 + 96\right) - \frac{f_0''\pi^2T^{-1}}{2(1+\epsilon^2)^3}\times \right. \left. \times\left(891\epsilon^8 + 28016\epsilon^6 + 82736\epsilon^4 + 43520\epsilon^2 + 3072\right)\right] . \tag{28}$$

In the next section we will go on to constraint the $f(R)$-theories estimating f_0'' from comparison between the theoretical predictions of $\dot{T}_b$ and the observed one.

4.1 Comparing theory prediction with data

It is well known that in the relativistic binary pulsar systems there is a loss of energy due to GWs emission. This energy loss, provided by GR, has been confirmed by the timing data analysis on the well known binary pulsar B1913+16 (Hulse and Taylor 1975; Weisberg et al. 2010). We also must note that the systems like B1913+16 are optimal tools to constrain theories of gravitation (Damour and Esposito-Farese 1998) using the Post-Keplerian parameters. For sake of convenience we choose the observed orbital period derivative $\dot{T}_b$, because it is one of the best observed Post-Keplerian parameters. Moreover we know, according to GR theory, that it is related to the foreseen orbital decay due to quadrupole gravitational radiation emitted by the binary systems.

As shown in sec. 4, it is possible rewrite the first derivative of the orbital period in $f(R)$-theories of gravity. In principle, if we know exactly which Lagrangian we have to use to describe those type of systems, then we can predict the energy loss through GWs radiation. Here, we want to make the inverse process, to get an estimation of the second derivative f_0'' imposing the strong hypothesis that the difference between the observed binary period variation ($\dot{T}_{b_{Obs}} \pm \delta$) and the one obtained by the relativistic theory of gravitation, $\Delta\dot{T}_b = \dot{T}_{b_{Obs}} - \dot{T}_{GR}$, is fully justified imposing that:

$$\dot{T}_{b_{Obs}} - \dot{T}_{GR} - f_0''\dot{T}_{b_{f(R)}} = 0, \tag{29}$$

$$\dot{T}_{b_{Obs}} \pm \delta - \dot{T}_{GR} - f_{0_{\pm\delta}}''\dot{T}_{b_{f(R)}} = 0, \tag{30}$$

and propagating the experimental error, $\pm\delta$, on the first derivative of the observed orbital period $\dot{T}_{b_{Obs}}$, into an uncertainty on second derivative of gravitational theory, $f_{0_{\pm\delta}}''$. What we want to emphasize is that, where GR is not able to fully explain the loss of energy by emission of GWs radiation then, the additive contribution of an ETGs can provide a way to fill the gap between theory and observations. We also have substracted the external contributions to the orbital decay as galactic or Shklovskii acceleration when those values are available in literature. Solving the eqs. (29) and (30) for f_0'' and $f_{0_{\pm\delta}}''$ we get an estimation of f_0'' and its upper and lower limits corresponding respectively to $\mp\delta$. In this way $\Delta\dot{T}_b$ is fully explained through the orbital period correction due to the ETGs $\dot{T}_{b_{f(R)}}$. So we get:

$$f_0'' = \frac{\Delta\dot{T}_b}{\dot{T}_{b_{f(R)}}}, \tag{31}$$

$$f_{0_{\pm\delta}}'' = \frac{\Delta\dot{T}_{b_{\pm\delta}}}{\dot{T}_{b_{f(R)}}}, \tag{32}$$

where $\Delta\dot{T}_{b_{\pm\delta}} = \dot{T}_{b_{Obs}} \pm \delta - \dot{T}_{GR}$.

Thus, among the various binary stars catalogues available in literature, we choose a sample of Observed Relativistic Binary Pulsars (ORBP) such that the binary period $T_{b_{Obs}}$, the observed orbital period variation $\dot{T}_{b_{Obs}}$, the computed orbital period variation from general relativistic theory $\dot{T}_{GR}$, the orbital eccentricity ϵ, the masses of the components m_p and m_c, are known with a fairly good precision. For each system we have chosen, all previous parameters and their references are reported in Tab. 1 where we show: the J-Name of the binary pulsar system, the observed orbital binary period $T_{b_{Obs}}$ in days, the orbital projected semi-major axis $asin(i)$ in light second, the orbital eccentricity ϵ, the observed time variation of the orbital period $\dot{T}_{b_{Obs}}$, the predicted one $\dot{T}_{GR}$ (according to the GR theory), the experimental error $\pm\delta$ on $\dot{T}_{b_{Obs}}$ and the masses m_p and m_c of the binary system components in solar mass unit.

Furthermore, in Tab. 2 we reported: the J-Name of the systems, the difference $\Delta\dot{T}_{GR}$ between $\dot{T}_{b_{Obs}}$ and $\dot{T}_{GR}$ (equal to the correction $-f_0''\dot{T}_{b_{f(R)}}$), the correction $\dot{T}_{b_{f(R)}}$, the corresponding f_0'' solution of (29) shown in (31), the interval centered on f_0'' and finally, the interval centered on f_0'' and computed from the difference: $\frac{f_{0_{+\delta}}'' - f_{0_{-\delta}}''}{2}$, where $f_{0_{\pm\delta}}''$, are the solutions of (29) shown in (31) taking in to account the experimental errors $\pm\delta$ on the observed orbital period variation $\dot{T}_{b_{Obs}}$.

Now, in Fig.1, we report representative results of our numerical analysis on the sample of binary pulsars we choose. In both panels we use the following notation: the black line shows the behavior of the first derivative of the orbital binary period for the $f(R)$-theories of gravity as computed in eq. (28); the blue line represents the observed orbital period variation $\dot{T}_{b_{Obs}}$; the red lines give the error band determined by the experimental errors $\pm\delta$; and finally the green line is representative of the $\dot{T}_{GR}$ orbital period variation computed from the GR. For the binary pulsar system $J2129 + 1210C$, Fig.1a, the orbital period variation $\dot{T}_{GR}$, computed from the GR, is included in experimental error band $\pm\delta$, so as the observed orbital period variation $\dot{T}_{b_{Obs}}$. Moreover, it is possible to see from Fig.1a the GR value of $\dot{T}_{GR}$ is recovered for $f_0'' = 0$ (green square), whilst to justify the difference $\Delta\dot{T}_{GR}$ between $\dot{T}_{b_{Obs}}$ and $\dot{T}_{GR}$ we have, from the solution of (31), the values shown in Fig.1 for $f_{0_{\pm\delta}}''$(red square) and for f_0'' (blue square). In the panel (b) of the Fig.1 there is reported for $J0751 + 1807$ the same situation. In this case the $\dot{T}_{GR}$ is out of the error band determined by the experimental errors $\pm\delta$. It is again possible to see for $f_0'' = 0$ that the GR value of $\dot{T}_{GR}$ is recovered, but in this case the f_0'' values are order of magnitude greater than the one of the well behaved case of $J2129 + 1210C$.

In Fig. 2 there are shown, for sake of convenience, in logarithmic scale, the absolute values of f_0'' reported in Tab.2 versus the ratio $\frac{\dot{T}_{b_{Obs}}}{\dot{T}_{GR}}$. We must note that for the first six binaries in tables, the ETGs are not ruled out $0.04 \leqslant f_0'' \leqslant 38$. For those systems we get $0.5 \leqslant \frac{\dot{T}_{b_{Obs}}}{\dot{T}_{GR}} \leqslant 1.5$, the difference between $\dot{T}_{GR}$ and $\dot{T}_{b_{Obs}}$ can be explained adding a new contribution from the theory of gravity. Instead for most of binaries we have f_0'' values that can surely rule out the theory, since taking account of the weak field assumption we obtain $38 \leqslant f_0'' \leqslant 4 \times 10^7$. From this last values to the first ones, there is a jump of about four up to five order of magnitude on f_0''. The origin of these strong discrepancies, perhaps, is due to the extreme assumption we made, to justify the difference between the observed $\dot{T}_{b_{Obs}}$ and the predicted $\dot{T}_{GR}$ using the ETGs.

5 DISCUSSION AND REMARKS

We want point out in this preliminary work that, where the GR theory is not enough to explain the gap between

the data and the theoretical estimation of the orbital decay, there is the possibility to extend the GR theory with a generic $f(R)$- theory to cover the gap. Here, we simply verify that this possibility exists, but there is need to compute the Post-Keplerian parameters in the $f(R)$-theory to estimate correctly the masses of the binary systems to constraint correctly the analytic parameters the ETGs. In post-Minkowskian limit of analytic $f(R)$-gravity models, the quadrupole-radiation depends on the masses of the two bodies, on the orbital parameters and on the analytic parameters of the $f(R)$-theory as the coefficients f_0' and f_0'' of the Taylor expansion. A first result we present is the analytical solution of the quadrupole radiation rate in which it is possible separate the GR contribution and the one due to the $f(R)$-gravity. We should note that the correction depends on the eccentricity of the orbit and on the orbital period of the binary system, and specifically, the radiation rate is a function of f_0' and f_0''. According to eq. (28),we have selected a sample of relativistic binary systems for which the first derivative of the orbital period is observed, we have computed the theoretical quadrupole radiation rate, and finally we have compared it to binary system observations. From Tab. 2, it is seen that the first five systems have masses determined in a manner quite reliable, while for the remaining sample, masses are estimated by requiring that the mass of the pulsar is $1.4M_\odot$ and, assuming for the orbital inclination one of the usual statistical values ($i = 60°$ or $i = 90°$), and from here comes then the estimate of the mass of the companion star. So a primary cause of major discrepancies, not only for the ETGs, but also for the GR theory, between the variation of the observed orbital period and the predicted effect of emission of gravitational waves, could be a mistake in the estimation of the masses of the system. In addition, other causes may be attributable to the evolutionary state of the system, which, for instance, if it does not consist of two neutron stars may transfer mass from companion to the neutron star. In our sample, there are only five double NS that can be used to test GR and ETGs. Taking into account of the strong hypothesis we made, the extended theory correction to $\dot{T}_{GR}$ can also include the galactic acceleration term correction (Damour and Taylor (1991), Damour and Taylor (1992)). Here, we give a preliminary result about the energy loss from binary systems and we show that, when the nature of the binary systems can exclude energy losses due to trade or loss of matter, then, we can explain the gap between the first time derivative of the observed orbital period and the theoretical one predicted by GR, using an analytical $f(R)$-theory of gravity. In conclusion, to improve the estimation of the $f(R)$-coefficients, we need: to consider the hydrodynamic effects due to the transfer of the matter in the binary system, in order to analyze different systems from double NS; and to improve the estimations of the mass of the stars in the binary systems without prior on pulsar mass and orbital inclination.

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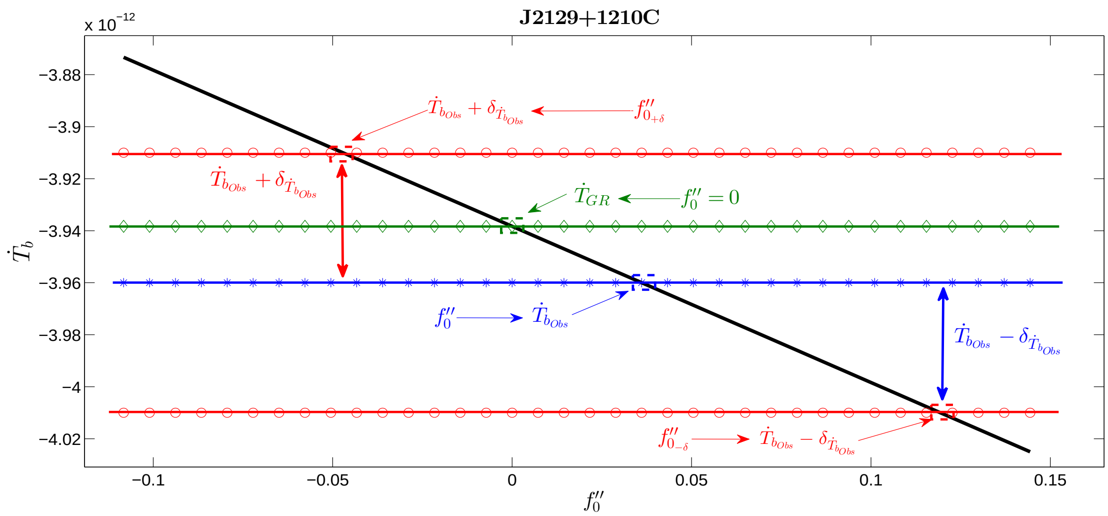
Table 1. Data for binary Relativistic pulsars: in the order we reported the J-Name of the binary pulsar system , the orbital binary period T_b in days, the orbital projected semi-major axis $a(sini)$ in light second, the orbital eccentricity ϵ, the Observed orbital period variation $\dot{T}_{Obs}$, the predicted $\dot{T}_{GR}$ according to the General Relativity theory, the experimental error $\pm\delta$ on $\dot{T}_{Obs}$ and the masses m_p and m_c of the binary system components in solar mass unit.

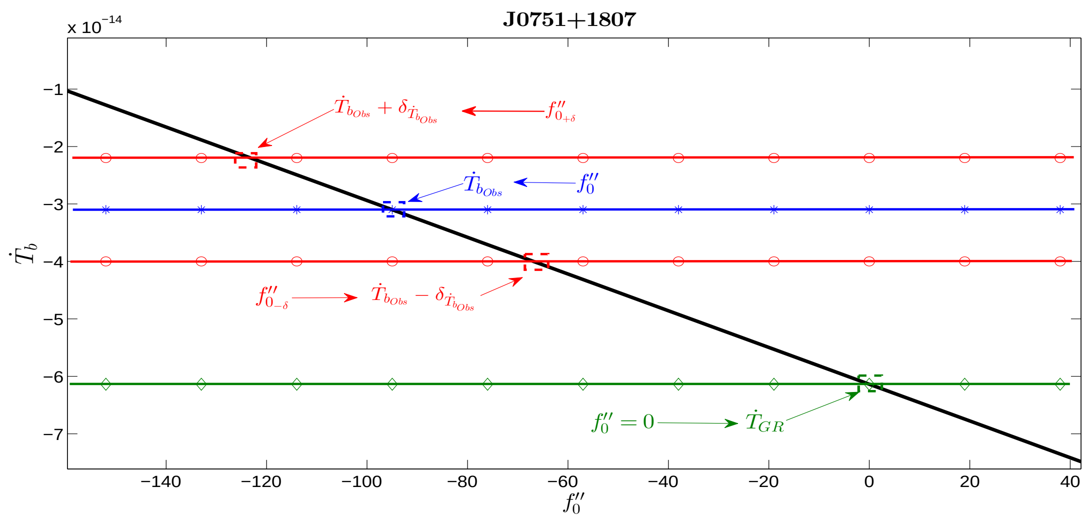
Name	T_b (days)	a (lsec)	i (degrees)	ϵ	$\dot{T}_{b_{Obs}}$ (10^{-12})	$\dot{T}_{GR}$ (10^{-12})	$\pm\delta$ (10^{-12})	m_p ($M_\odot$)	m_c ($M_\odot$)	**References**
J2129+1210C	0.335282049	2.51845		0.681395	-3.96	-3.94	0.05	1.358	1.354	Anderson et al. (1990), Jacoby et al. (2006)
J1915+1606	0.322997449	2.341782	54.12°	0.6171334	-2.423	-2.403	0.001	1.4398	1.3886	Hulse and Taylor (1975),Weisberg et al. (2010)
J0737-3039A	0.102251562	1.415032	88.69°	0.0877775	-1.252	-1.248	0.017	1.3381	1.2489	Burgay et al. (2003), Kramer et al. (2006)
J1141-6545	0.197650959	1.858922	73°	0.171884	-0.403	-0.387	0.025	1.27	1.02	Kaspi et al. (2000); Bhat et al. (2008)
J1537+1155	0.420737299	3.7294626	78.4°	0.2736767	-0.138	-0.192	0.0001	1.3332	1.3452	Stairs et al. (2002); Konacki et al. (2003)
J1738+0333	0.3547907399	0.343429	32.6°	3.4e-7	-0.017	-0.0277	0.0031	1.46	0.181	Freire et al. (2012)
J0751+1807	0.263144267	0.3966127	65.8°	0.00000071	-0.031	-0.017	0.009	1.7	0.67	Lundgren et al. (1995); Nice et al. (2008)
J0024-7204J	0.120664938	0.0404021	60°	0	-0.55	-0.03	0.13	1.4	0.024	Freire et al. (2003); Camilo et al. (2000)
J1701-3006B	0.144545417	0.2527565	84.7°	0	-5.12	-0.09	0.062	1.4	0.14	Possenti et al. (2003); Lynch et al. (2012)
J2051-0827	0.099110251	0.045052	30°	0	-15.5	-0.03	0.8	1.4	0.027	Stappers et al. (1996); Doroshenko et al. (2001)
J1909-3744	1.533449475	1.8979910	86.4°	1.302E-07	-0.55	-0.003	0.03	1.57	0.212	Jacoby et al. (2003); Verbiest et al. (2009)
J1518+4904	8.634005096	20.044002	$< 47°$	0.24948451	0.24	-0.001	0.22	1.56	1.05	Nice et al. (1996); Janssen et al. (2008)
J1959+2048	0.381966607	0.0892253	65°	0	14.7	-0.003	0.8	1.4	0.022	Fruchter et al. (1988); Arzoumanian et al. (1994)
J2145-0750	6.83893	10.164108		0.0000193	0.4	-0.0005	0.3	1.4	0.5	Bailes et al. (1994); Verbiest et al. (2009)
J0437-4715	5.74104646	3.36669708	137.58°	0.00001918	0.159	-0.0004	0.283	1.76	0.254	Johnston et al. (1993); Verbiest et al. (2008)
J0045-7319	51.169451	174.2576	44°	0.807949	-3.03E+5	-0.02242	9E+3	1.4	8.8	McConnell et al. (1991); Kaspi et al. (1996)
J2019+2425	76.51163479	38.7676297	63°	0.00011109	-30.0	-0.000006	60.0	1.33	0.35	Nice et al. (1993, 2001)
J1623-2631	191.44281	64.80946	40°	0.02531545	400.0	-0.000003	600.0	1.3	0.8	Lyne et al. (1988); Thorsett et al. (1999)

Table 2. Upper Limits of f_0'' correction to $\dot{T}_{GR}$ of binary relativistic pulsars assuming that all the loss of energy is caused by Gravitational Wave emission. We reported the J-Name of the system,the difference $\Delta\dot{T}_{GR}$ between $\dot{T}_{b_{Obs}}$ and $\dot{T}_{GR}$ equal to the correction $-f_0''\dot{T}_{b_{f(R)}}$, the correction $\dot{T}_{b_{f(R)}}$, the corresponding f_0'' solution of (29) shown in (31), the interval centered on f_0'' and computed from the difference $\frac{f''_{0+\delta}-f''_{0-\delta}}{2}$,where $f''_{0\pm\delta}$, are the solutions of (29) shown in (31) taking account of the experimental errors $\pm\delta$ on the observed orbital period variation $\dot{T}_{b_{Obs}}$.

Name	$\Delta\dot{T}_{GR}$	$\dot{T}_{b_{f(R)}}$	f_0''	$\pm\Delta f_0''$
J2129+1210C	-2.17E-14	6.01E-13	3.61E-02	8.32E-02
J1915+1606	-2.04E-14	2.10E-13	9.74E-02	4.77E-03
J0737-3039A	-4.23E-15	1.86E-14	2.28E-01	9.15E-02
J1141-6545	-1.65E-14	3.88E-15	4.25E+00	6.44E+00
J1537+1155	5.39E-14	1.42E-15	-3.79E+01	7.03E-02
J1738+0333	-1.56E-15	1.06E-16	-1.47E+01	2.92E+01
J0751+1807	1.41E-13	8.98E-16	-15.7E+01	1.002E+01
J0024-7204J	-5.22E-13	3.13E-16	1.67E+03	4.15E+02
J1701-3006B	-5.03E-12	8.81E-16	5.71E+03	7.04E+01
J2051-0827	-1.55E-11	4.77E-16	3.24E+04	1.68E+03
J1909-3744	-5.47E-13	2.62E-18	2.09E+05	1.14E+04
J1518+4904	2.41E-13	3.42E-19	-7.05E+05	6.43E+03
J1959+2048	1.47E-11	1.07E-17	-1.38E+06	7.51E+04
J2145-0750	4.01E-13	1.00E-19	-4.00E+06	2.99E+06
J0437-4715	1.59E-13	1.04E-19	-1.57E+06	2.73E+06
J0045-7319	3.02E-07	1.11E-16	2.74E+9	8.13E+07
J2019+2425	-3.00E-11	1.11E-22	2.71E+11	5.41E+11
J1623-2631	4.00E-10	2.02E-23	-1.98E+13	2.97E+13



(a)



(b)

Figure 1. We report representative results of our numerical analyses on the sample of binary pulsars we have selected. In both figures we use the following notation: the black line shows the behavior of the first derivative of the orbital binary period for the $f(R)$-theory of gravitation as computed in eq. (28); the blue line represents the observed orbital period variation $\dot{T}_{b_{Obs}}$; the red lines give the error band determined by the experimental errors $\pm\epsilon$; and finally the green line is representative of the $\dot{T}_{GR}$ orbital period variation computed from the GR. In the panel (a) for the system **J2129+1210 C** the $\dot{T}_{GR}$ is included in the error band determined by the experimental errors $\pm\delta$, so as $\dot{T}_{b_{Obs}}$. We point out the GR value of $\dot{T}_{GR}$ is recovered for $f''(r) = 0$ (green square), while to justify the difference between $\dot{T}_{b_{Obs}}$ and $\dot{T}_{GR}$ we show the value of f''_0 (blue square) and its error band $f''_{0\pm\delta}$ (red square) as computed in eqs. (31) and (32). In the last panel (b) there are reported for **J0751+1807** the same data but in this case the $\dot{T}_{GR}$ is **OUT** of the error band determined by the experimental errors $\pm\delta$. It is possible to see for $f''_0 = 0$ that the GR value of $\dot{T}_{GR}$ is recovered, but in this case the f''_0 values are much greater than the previous ones.

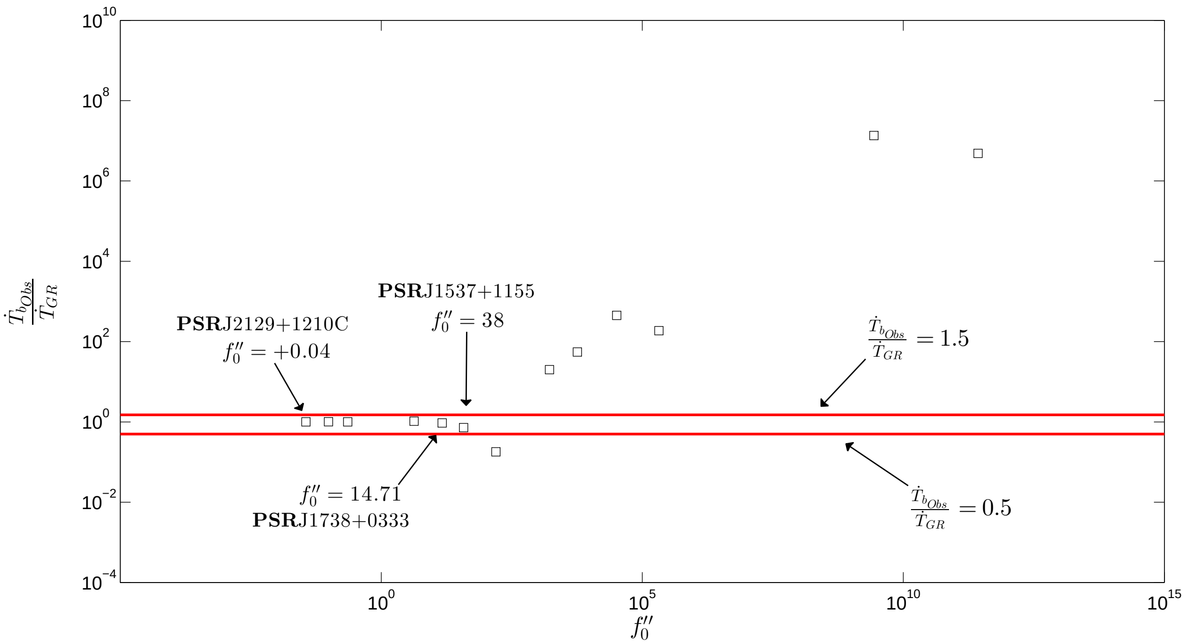


Figure 2. In figure there are shown, for sake of convenience, in logaritmic scale, the absolute values of f_0'' reported in Tab. 2 versus the ratio $\frac{\dot{T}_{b_{Obs}}}{\dot{T}_{GR}}$. We must note that for five binaries the ETGs we are probing is not ruled out $0.04 \leqslant f_0'' \leqslant\approx 38$, for those systems the difference between $\dot{T}_{GR}$ and $\dot{T}_{b_{Obs}}$ is tiny, indeed we get $0.5 \leqslant \frac{\dot{T}_{b_{Obs}}}{\dot{T}_{GR}} \leqslant 1.5$. Instead for most of binaries we have f_0'' values that can surely rule out the theory, since taking account of the weak field assumption we obtain $38 \leqslant f_0'' \leqslant 4 \times 10^7$. From this last values to the first ones, there is a jump of about four up to five order of magnitude on f_0''.